Relativistic Quantum Transport Theory for Electrodynamics[*]

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Abstract

We investigate the relationship between the covariant and the three-dimensional (equal-time) formulations of quantum kinetic theory. We show that the three-dimensional approach can be obtained as the energy average of the covariant formulation. We illustrate this statement in scalar and spinor QED. For scalar QED we derive Lorentz covariant transport and constraint equations directly from the Klein-Gordon equation rather than through the previously used Feshbach-Villars representation. We then discuss the relation to their equal-time version. We make a semiclassical expansion in $\hbar$ and obtain Vlasov-type transport equations for the zeroth and first order of the scalar Wigner operator. We then consider pair production in a spatially homogeneous but time-dependent electric field and show that the pair density is derived much more easily via the energy averaging method than in the equal-time representation. Proceeding to spinor QED, we derive the covariant version of the equal-time equation derived by Bialynicki-Birula et al. We show that it must be supplemented by another self-adjoint equation to obtain a complete description of the covariant spinor Wigner operator. After spinor decomposition and energy average we study the classical limit of the resulting three-dimensional kinetic equations. There are only two independent spinor components in this limit, the mass density and the spin density, and we derive also their covariant equations of motion. We then show that the equal-time kinetic equation provides a complete description only for constant external electromagnetic fields, but is in general incomplete. It must be supplemented by additional constraints which we derive explicitly from the covariant formulation.

1   Introduction

Theoretic evidence from lattice simulations [1] of QCD for a phase transition from a hadron gas to a quark-gluon plasma at high temperature has prompted experimental efforts to create this new phase of matter in the laboratory during the early stages of ultrarelativistic heavy ion collisions [2]. However, because of the estimated very short life time of the collision zone, the highly excited quark-gluon system may spend a considerable fraction of its life in a non-thermalized, pre-equilibrium state. The dynamical tool to treat dissipative processes in heavy ion collisions and the approach to local thermal equilibrium is in principle non-equilibrium quantum transport theory. Since QCD, the theory describing the interactions between quarks and gluons, is a gauge theory, the kinetic equations should be gauge covariant. A relativistic and gauge covariant kinetic theory for quarks and gluons has been derived [3], both in a classical framework [4, 5] and as a quantum kinetic theory [6, 7] based on Wigner operators defined in 8-dimensional phase space [8]. Some preliminary applications to the quark-gluon plasma, such as linear color response, color correlations and collective plasma oscillations [9, 10] have been discussed in this framework using a semiclassical expansion of the quantum transport theory. Apart from its relativistic and gauge covariance, an important aspect of the four-dimensional approach to transport theory is that the complex kinetic equation can be split up into a constraint and a transport equation, where the former is a quantum extension of the classical mass-shell condition, and the latter is a covariant generalization of the Vlasov-Boltzmann equation. The complementarity of these two ingredients is essential for a physical understanding of quantum kinetic theory [11].

As a first step towards a full kinetic treatment of the quark-gluon plasma, a detailed study of relativistic transport theory for electrodynamics was performed over the years (see [3, 12] for references). The absence of the complications arising from the non-abelian character of QCD allows for a deeper insight into the structure of the kinetic theory itself, and it is also easier to find useful applications in this simpler case. Vasak, Gyulassy and Elze [13] (to be quoted as VGE) discussed the relativistic quantum transport

theory for spinor electrodynamics and gave a very lucid account of the covariant spinor decomposition for the Wigner operator of Dirac particles (see also [12]).

In classical transport theory, all the physical currents are connected with the distribution function f. The quantum mechanical analogue of f is the Wigner function, which is the ensemble expectation of the Wigner operator. In the process of performing the ensemble average of the kinetic equation for the Wigner operator, one encounters the two-body correlation function $\langle F^{\mu\nu}\hat{W}\rangle$, where $F^{\mu\nu}$ is the gauge field strength. In the general case, the two-body correlation function again depends on higher order correlation functions; this generates the so called BBGKY hierarchy [8]. A popular method to obtain a closed kinetic equation for the Wigner function $W = \langle\hat{W}\rangle$ is to use the Hartree approximation where the gauge field is considered as a mean field $\bar{F}^{\mu\nu}$, leading to the replacement $\langle F^{\mu\nu}\hat{W}\rangle = \bar{F}^{\mu\nu}\langle\hat{W}\rangle$. In this approximation the BBGKY chain is truncated at the one body level, and the kinetic equation for the Wigner function has the same form as that for the Wigner operator. This version of the mean field approximation, which treats the particle fields quantum mechanically, but uses the classical approximation for the gauge fields, is widely used in electrodynamic transport theory [13, 14, 15]. It is appropriate for strong but slowly varying electromagnetic fields. An interesting example is the Schwinger process [16, 17] for pair production in external electromagnetic fields. Since this approximation is sufficient for our purpose, we will also restrict ourselves to the investigation of transport theory for particles interacting with an external field.

In a recent paper Bialynicki-Birula, Gornicki and Rafelski [14] (to be quoted as BGR) proposed an equal-time transport equation for spinor electrodynamics. Unlike the covariant theory where the Wigner operator is defined as the four-dimensional Fourier transform of the gauge covariant density matrix $\Phi_4(x, y)$ (see below),

$$\hat{W}_4(x, p) = \int d^4 y \, e^{ip\cdot y} \, \Phi_4(x, y) \,, \tag{1}$$

they introduced an equal-time correlation function $\Phi_3(x, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{y}, t)$ and considered its spatial Fourier transform which depends only on three momentum coordinates:

$$\hat{W}_3(x, \mathbf{p}) = \int d^3 y \, e^{-i\mathbf{p}\cdot\mathbf{y}} \, \Phi_3(x, \mathbf{y}) \,. \tag{2}$$

From this definition and the Dirac equation they obtained directly a kinetic equation for $\hat{W}_3$. They showed that the Wigner function $\langle \hat{W}_3(x, \mathbf{p}) \rangle$ is a direct analogue of the classical distribution function $f(\mathbf{x}, \mathbf{p}, t)$ and that it provides a systematic way of studying the phase-space dynamics of QED in the semiclassical limit; each spinor component of $\langle \hat{W}_3 \rangle$ corresponds to a definite physical distribution function. Of course, $\hat{W}_3(x, \mathbf{p})$ is not manifestly Lorentz covariant.

Recently this equal-time method has been extended [15] to scalar electrodynamics in the Feshbach-Villars representation. The ensuing applications of this so-called "three-dimensional" transport theory to the problem of nonperturbative pair creation in both spinor [14] and scalar [18, 19] QED indicated that some quantum problems may be treated more easily in phase space in terms of the Wigner operator than by conventional field theoretic methods. We will show, however, that the published derivation of the equal-time transport theory is incomplete. A complete treatment leads to additional constraints on the Wigner function which for the specific situations treated by previous authors actually simplifies its structure; in general, however, it leads to complications whose effects cannot be neglected.

We show in this paper that the three-dimensional transport theory can also be derived by a different method which does not suffer from this incompleteness. In particular, we discuss the relationship between the covariant and three-dimensional approaches. From the comparison of Eqs.(1) and (2) it is easily seen that there exists a general connection between the four- and three-dimensional Wigner operators:

$$\hat{W}_3(x, \mathbf{p}) = \int dp_0 \, \hat{W}_4(x, p) \, . \tag{3}$$

This relation shows that the three-dimensional transport theory can be obtained by an energy average of its covariant version. The main formal difference between the energy averaging method used in this paper and the direct equal-time derivation of BGR is that our approach starts with an explicitly relativistic formulation. This covariant formulation is shown to be complete, i.e. the equation of motion for the Wigner operator is

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equivalent to the Dirac equation for the field operators. This completeness is preserved during the reduction from the four- to the three-dimensional version via energy averaging. The covariant formulation followed by energy averaging also allows us to obtain a three-dimensional transport equation for the scalar Wigner operator directly from the Klein-Gordon equation, rather than via the Feshbach-Villars 2×2 matrix formulation.

We proceed as follows. In section 2, we discuss the transport theory for spinless charged particles. We first set up the covariant constraint and transport equations, which are the basis of the reduction to the three-dimensional version. Next we consider the semiclassical expansion in $\hbar$ and the classical limit. Then we average the covariant equations over the energy to obtain the three-dimensional equations. With these we investigate the pair creation of charged scalar particles in a spatially homogeneous but time-dependent electric field. This problem was studied before in the Feshbach-Villars matrix representation [19], but our procedure turns out to be much simpler because of the scalar nature of our Wigner function. In Section 3, we study the kinetic theory for Dirac fermions. We derive a complete set of two selfadjoint covariant equations, one of which corresponds to the four-dimensional version of the BGR equation. We give a full mapping of the spinor components from the four- to the three-dimensional representation. The two selfadjoint covariant transport equations can be combined into a single complex equation, the VGE equation, which is then subjected to the energy averaging procedure. We discuss the classical limit of the resulting three-dimensional equations and compare it with the results [20] from the equal-time method. We then show how to get a complete set of three-dimensional kinetic equations in the general, fully quantum mechanical case; these contain as a subset the various spinor components of the BGR equation, but also a set of additional constraints. We discuss these results and summarize them in our conclusions.

2 Scalar electrodynamics

2.1 Relativistic covariant kinetic equations

The abelian gauge theory of scalar particles with mass m and charge e is defined through the Lagrangian density

$$\mathcal{L} = (\partial_\mu - ieA_u)\phi^\dagger(\partial^\mu + ieA^\mu)\phi - m^2\phi^\dagger\phi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \ . \tag{4}$$

It leads to the Klein-Gordon equation for the scalar field operator ϕ,

$$\left[(\partial_\mu + ieA_\mu(x))(\partial^\mu + ieA^\mu(x)) + m^2\right]\phi(x) = 0 \ , \tag{5}$$

the adjoint equation for $\phi^\dagger$, and to Maxwell's equations for the field strength tensor $F^{\mu\nu}$,

$$\partial_\mu F^{\mu\nu}(x) = j^\nu(x) \ , \qquad \partial_\mu \tilde{F}^{\mu\nu}(x) = 0 \ , \tag{6}$$

where $\tilde{F}^{\mu\nu}(x) = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual field tensor and $j^\nu(x)$ is the charge current of the scalar particles.

The relativistically covariant Wigner operator is the Fourier transform Eq. (1) of the gauge covariant scalar field correlation function

$$\Phi_4(x, y) = \phi(x)e^{-y\cdot\mathcal{D}^\dagger/2} \cdot e^{y\cdot\mathcal{D}/2}\phi^\dagger(x) \ . \tag{7}$$

The covariant derivatives $\mathcal{D}_\mu(x) = \partial_\mu + ieA_\mu(x)$ and $\mathcal{D}_\mu^\dagger(x) = \partial_\mu - ieA_\mu(x)$ in this expression ensure the gauge covariance (actually: gauge independence in the case of a $U(1)$ gauge theory) of the density matrix. Following [6] and replacing the covariant derivatives here by a line integral of the gauge field $A_\mu(x)$ along a straight line between the space-time points $x - \frac{y}{2}$ and $x + \frac{y}{2}$, we recover the more familiar form [16]

$$\Phi_4(x, y) = \phi\left(x + \tfrac{1}{2}y\right) \exp\left[ie \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, A(x + sy){\cdot}y\right] \phi^\dagger\left(x - \tfrac{1}{2}y\right) \ . \tag{8}$$

The resulting scalar Wigner operator is selfadjoint:

$$\hat{W}_4^\dagger(x, p) = \hat{W}_4(x, p) \ . \tag{9}$$

Substituting Eq. (7) into Eq. (1) and integrating over y we see that the Wigner operator is just the density of particles at space-time point x with kinetic momentum p,

$$\hat{W}_4(x, p) = \phi(x)\, \delta^4(p - \hat{\pi}(x))\, \phi^\dagger(x)\,, \tag{10}$$

where $\hat{\pi}(x) = \frac{i}{2}(\mathcal{D}_\mu(x) - \mathcal{D}_\mu^\dagger(x))$ is the kinetic momentum operator. Eq. (10) motivates a statistical interpretation of the Wigner function as a generalized phase-space density and generates simple relations between the Wigner operator and all physical space-time density operators. For example, the operators for the charge current density and the energy momentum tensor are given by

$$
\begin{aligned}
\hat{j}_\mu(x) &= ie\Big(\phi^\dagger(x)\partial_\mu\phi(x) - (\partial_\mu\phi^\dagger(x))\phi(x)\Big) \\
&= e\int d^4p\, p_\mu\, \hat{W}_4(x, p)\,, \\
\hat{T}_{\mu\nu}(x) &= \tfrac{1}{2}\Big((\partial_\mu\phi^\dagger(x))(\partial_\nu\phi(x)) - (\partial_\nu\phi^\dagger(x))(\partial_\mu\phi(x))\Big) \\
&= \int d^4p\, p_\mu p_\nu\, \hat{W}_4(x, p)\,.
\end{aligned}
\tag{11}
$$

The equations of motion for the Wigner operator are a direct consequence of the field equations (5). We calculate the second-order derivatives of the correlator $\Phi_4(x, y)$ with respect to x and y,

$$
\begin{aligned}
\left(\tfrac{1}{2}\partial_\mu^x + \partial_\mu^y\right)\left(\tfrac{1}{2}\partial_x^\mu + \partial_y^\mu\right)\Phi_4(x, y) =\ & -m^2\Phi_4(x, y) \\
& -2ie\int_{-\frac{1}{2}}^{\frac{1}{2}} ds\left(\tfrac{1}{2} + s\right) y_\nu F^{\nu\mu}(x + sy)\left(\tfrac{1}{2}\partial_\mu^x + \partial_\mu^y\right)\Phi_4(x, y) \\
& -(ie)^2\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} ds\left(\tfrac{1}{2} + s\right) y{\cdot}F(x + sy)\right]^2\Phi_4(x, y) \\
& +ie\int_{-\frac{1}{2}}^{\frac{1}{2}} ds\left(\tfrac{1}{2} + s\right)^2 y{\cdot}j(x + sy)\, \Phi_4(x, y)\,,
\end{aligned}
\tag{12}
$$

where we have employed the Klein-Gordon and Maxwell equations in the first and last terms on the right-hand side. Eq. (12) can be rewritten in a compact way,

$$\left[\tfrac{1}{4}D^\mu D_\mu - \Pi^\mu\Pi_\mu + m^2 - i\Pi^\mu D_\mu\right]\Phi_4(x, y) = 0\,, \tag{13}$$

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where we defined the two Lorentz covariant operators D and Π by

$$
\begin{aligned}
D^\mu(x,y) &= \partial_x^\mu + ie \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, y_\nu F^{\nu\mu}(x+sy)\,, \\
\Pi^\mu(x,y) &= i\left(\partial_y^\mu + ie \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, s\, y_\nu F^{\nu\mu}(x+sy)\right)\,.
\end{aligned}
\tag{14}
$$

Performing the Fourier transform with respect to y, we obtain the following exact quadratic kinetic equation for the scalar Wigner operator:

$$
\left[\tfrac{1}{4}\hbar^2 D^\mu D_\mu - \Pi^\mu \Pi_\mu + m^2 - i\hbar \Pi^\mu D_\mu\right]\hat{W}_4(x,p) = 0\,.
\tag{15}
$$

The operators D and Π in this equation are now defined in phase space and can be obtained from Eqs. (14) by the replacements $y^\mu \to -i\partial_p^\mu$ and $\partial_y^\mu \to -ip^\mu$:

$$
\begin{aligned}
D_\mu(x,p) &= \partial_\mu - e \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, F_{\mu\nu}(x - i\hbar s\partial_p)\, \partial_p^\nu\,, \\
\Pi_\mu(x,p) &= p_\mu - ie\hbar \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, s\, F_{\mu\nu}(x - i\hbar s\partial_p)\, \partial_p^\nu\,.
\end{aligned}
\tag{16}
$$

In Eqs. (15) and (16) we reinstated the $\hbar$-dependence explicitly in order to be able to discuss the semiclassical expansion in the next subsection. (The speed of light c is still omitted.) Obviously, the operators D_μ and Π_μ are gauge covariant extensions of the partial derivative ∂_μ and the momentum p_μ, respectively. They both are self-adjoint: namely, $D_\mu^\dagger(x,p) = D_\mu(x,p)$ and $\Pi_\mu^\dagger(x,p) = \Pi_\mu(x,p)$.

Since the scalar Wigner operator is self-adjoint, too, the real and imaginary parts of the complex equation (15) have to vanish separately:

$$
\left(\tfrac{1}{4}\hbar^2 D^\mu D_\mu - \Pi^\mu \Pi_\mu + m^2\right)\hat{W}_4(x,p) = 0\,,
\tag{17}
$$

$$
\hbar \Pi^\mu D_\mu \hat{W}_4(x,p) = 0\,.
\tag{18}
$$

The first equation is usually called constraint equation since it is obviously a generalization of the classical mass-shell condition $p^2 - m^2 = 0$. The second equation has the typical form of a transport equation and is the generalization of the classical Vlasov equation for charged particles with abelian interactions $p^\mu(\partial_\mu - eF_{\mu\nu}(x)\partial_p^\nu)f(x,p) = 0$, where $f(x,p)$ is the classical distribution function. These two equations together give a complete description for the Wigner operator; they are equivalent to the original field equations of motion.

2.2 Semiclassical expansion

In order to better understand the structure of the constraint and transport equations, we consider their semiclassical expansion in $\hbar$ and their classical limit. The calculation of quantum corrections to this limit can then be performed in a systematic way.

As stated in the Introduction, we will in this paper consider the electromagnetic field as a classical (mean) field. In this approximation the equation of motion for the Wigner *operator* and its ensemble expectation value, the Wigner *function*, become formally identical. Therefore we will now work with the equations of motion for the Wigner *function* by leaving off the hats over the corresponding operators.

The field strength $F^{\mu\nu}$ in Eqs. (16), which has to be evaluated at the shifted argument $x - i\hbar s \partial_p$, is defined in terms of its Taylor expansion around x and can be expressed in terms of the so-called "triangle operator" $\triangle = \partial_p \cdot \partial_x$ as

$$F_{\mu\nu}(x - is\hbar\partial_p) = e^{is\hbar\triangle} F_{\mu\nu}(x) \,. \tag{19}$$

The s-integration can then be done, and expanding the result in powers of $\hbar$ we obtain

$$
\begin{aligned}
D_\mu(x,p) &= \partial_\mu - e \frac{\sin(\hbar\triangle/2)}{\hbar\triangle/2} F_{\mu\nu}(x)\, \partial_p^\nu \\
&= \left(\partial_\mu - e F_{\mu\nu}(x)\, \partial_p^\nu\right) + \frac{e}{24}\hbar^2 \triangle^2 F_{\mu\nu}(x)\, \partial_p^\nu + \cdots \,, \\
\Pi_\mu(x,p) &= p_\mu + \frac{e\hbar}{2}\left(\frac{\cos(\hbar\triangle/2)}{\hbar\triangle/2} - \frac{\sin(\hbar\triangle/2)}{(\hbar\triangle/2)^2}\right) F_{\mu\nu}(x)\, \partial_p^\nu \\
&= p_\mu - \frac{e}{12}\hbar^2 \triangle F_{\mu\nu}(x)\, \partial_p^\nu + \cdots \,,
\end{aligned}
\tag{20}
$$

where the dots indicate corrections from higher orders of $\hbar$ or, equivalently, of the derivative operator $\triangle$.

Expanding the Wigner function similarly in powers of $\hbar$,

$$W_4(x,p) = W_4^{(0)}(x,p) + \hbar W_4^{(1)}(x,p) + \hbar^2 W_4^{(2)}(x,p) + \cdots \,, \tag{21}$$

and inserting these expressions into the constraint and transport equations (17) and (18), we obtain their semiclassical expansion. In the zeroth order, there is no information from

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the transport equation, and the constraint equation reduces to the classical mass-shell condition,

$$(p^2 - m^2) W_4^{(0)}(x, p) = 0 . \tag{22}$$

This equation has two elementary solutions corresponding to positive and negative energies, and we can write

$$W_4^{(0)}(x, p) = W_4^{+(0)}(x, \mathbf{p}) \, \delta(p_0 - E_p) + W_4^{-(0)}(x, \mathbf{p}) \, \delta(p_0 + E_p) , \tag{23}$$

where $E_p = +\sqrt{\mathbf{p}^2 + m^2}$ is the classical on-shell energy.

To next order in $\hbar$, the transport equation (18) begins to contribute. It yields the Vlasov equation

$$p^\mu \left(\partial_\mu - e F_{\mu\nu}(x) \partial_p^\nu \right) W_4^{(0)}(x, p) = 0 , \tag{24}$$

which must be evaluated with the ansatz (23). The constraint equation yields at order $\hbar$

$$(p^2 - m^2) W_4^{(1)}(x, p) = 0 , \tag{25}$$

with the solution

$$W_4^{(1)}(x, p) = W_4^{+(1)}(x, \mathbf{p}) \, \delta(p_0 - E_p) + W_4^{-(1)}(x, \mathbf{p}) \, \delta(p_0 + E_p) . \tag{26}$$

At order $\hbar^2$, Eq. (17) becomes

$$(p^2 - m^2) W_4^{(2)}(x, p) = \left[\frac{1}{4} \left(\partial_x - e F(x) \cdot \partial_p \right)^2 + \frac{e}{12} \left(p^\mu \triangle F_{\mu\nu} \, \partial_p^\nu + j(x) \cdot \partial_p^\nu \right) \right] W_4^{(0)}(x, p) , \tag{27}$$

where $j_\nu(x) = e \int d^4p \, p_\nu \, W(x, p)$ is the average charge current of the scalar particles in the ensemble which generates the mean electromagnetic field via Maxwell's equation (6). Eq. (18) yields at order $\hbar^2$

$$p^\mu \left(\partial_\mu - e F_{\mu\nu}(x) \partial_p^\nu \right) W_4^{(1)}(x, p) = 0 . \tag{28}$$

The right-hand side of (27) represents off-shell corrections to the mass-shell condition due to quantum corrections. Eq. (28) is the Vlasov equation for the first order Wigner operator. Quantum corrections to the Vlasov equation for $W_4^{(2)}$ will arise only at order

$\hbar^3$. The fact that quantum corrections affect the Wigner function only at second order in $\hbar$ is a specific feature of scalar particles. In spinor QED quantum effects arise in first order of $\hbar$ due to spin interactions.

We now carry out the p_0-average of the Vlasov equations (24,28). This is facilitated by their simple form (23,26) resulting from the mass-shell conditions (22,25). For the corresponding three-dimensional positive and negative energy Wigner functions $W_3^{\pm}$ (see Eq. (3)) we obtain

$$\partial_t W_3^{\pm(i)}(x, \mathbf{p}) \pm (\mathbf{v}\cdot\nabla)W_3^{\pm(i)}(x, \mathbf{p}) + e\big(\mathbf{E}(x) \pm \mathbf{v} \times \mathbf{B}(x)\big)\cdot\nabla_p W_3^{\pm(i)}(x, \mathbf{p}) = 0 \,, \quad (29)$$

where $\mathbf{v} = \mathbf{p}/E_p$, $i = 0, 1$, and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field components of $F^{\mu\nu}$. We now introduce classical particle and antiparticle distribution functions, f and $\bar{f}$, via

$$\begin{aligned}
f^{(i)}(x, \mathbf{p}) &= W_3^{+(i)}(x, \mathbf{p}) \,, \\
\bar{f}^{(i)}(x, \mathbf{p}) &= W_3^{-(i)}(x, -\mathbf{p}) \,.
\end{aligned} \quad (30)$$

They satisfy the well-known three-dimensional Vlasov transport equations for scalar particles and antiparticles moving in an external electromagnetic field:

$$\begin{aligned}
\partial_t f^{(i)}(x, \mathbf{p}) + (\mathbf{v}\cdot\nabla)f^{(i)}(x, \mathbf{p}) + e\big(\mathbf{E}(x) + \mathbf{v} \times \mathbf{B}(x)\big)\cdot\nabla_p f^{(i)}(x, \mathbf{p}) = 0 \,, \\
\partial_t \bar{f}^{(i)}(x, \mathbf{p}) + (\mathbf{v}\cdot\nabla)\bar{f}^{(i)}(x, \mathbf{p}) - e\big(\mathbf{E}(x) + \mathbf{v} \times \mathbf{B}(x)\big)\cdot\nabla_p \bar{f}^{(i)}(x, \mathbf{p}) = 0 \,.
\end{aligned} \quad (31)$$

2.3 Pair production in transport theory

In this subsection, we give an exact non-perturbative solution of the scalar transport theory in a spatially constant external electric field. The solution describes pair production due to vacuum excitation. In electrodynamics, it was first studied many years ago by Schwinger [16], who connected the probability of pair creation with the imaginary part of the effective action in QED. Recently it was reinvestigated as an application of the equal-time transport theory. For scalar electrodynamics in the Feshbach-Villars representation, Best and Eisenberg [19] obtained from the kinetic theory the same result as Popov did previously in [21] with field operator techniques. Here we consider the pair creation of scalar particles directly in the Klein-Gordon representation of the kinetic theory, to show how the energy averaging method works in a non-perturbative case and to discuss its difference from the Feshbach-Villars based equal-time method.

2.3.1 Initial value problem

The determination of any solution of the differential equations (17) and (18) needs initial conditions. For the pair creation problem we should search for a free vacuum solution as the initial condition. For $\mathbf{E} = \mathbf{B} = 0$, the energy average of the full constraint and transport equations result in the following three-dimensional expressions:

$$
\begin{aligned}
\left(\partial_t^2 - \nabla^2 + 4E_p^2\right)W_3(x, \mathbf{p}) &= 4\,\varepsilon(x, \mathbf{p})\,, \\
\partial_t\,\rho(x, \mathbf{p}) - \nabla\!\cdot\!\mathbf{j}(x, \mathbf{p}) &= 0\,.
\end{aligned}
\tag{32}
$$

Here we defined the phase-space densities of electric charge $\rho(x, \mathbf{p})$, electric current $\mathbf{j}(x, \mathbf{p})$, and energy $\varepsilon(x, \mathbf{p})$ by

$$
\begin{aligned}
\rho(x, \mathbf{p}) &= \int dp_0\, j_0(x, p) = e\int dp_0\, p_0\, W_4(x, p)\,, \\
\mathbf{j}(x, \mathbf{p}) &= \int dp_0\, \mathbf{j}(x, p) = e\,\mathbf{p}\, W_3(x, \mathbf{p})\,, \\
\varepsilon(x, \mathbf{p}) &= \int dp_0\, T_{00}(x, p) = \int dp_0\, p_0^2\, W_4(x, p)\,.
\end{aligned}
\tag{33}
$$

The simplest homogeneous solution of Eq. (32) is

$$
\begin{aligned}
W_3(x, \mathbf{p}) &= \frac{1}{E_p}\,, \\
\rho(x, \mathbf{p}) &= 0\,, \\
\mathbf{j}(x, \mathbf{p}) &= e\mathbf{v} = e\frac{\mathbf{p}}{E_p}\,, \\
\varepsilon(x, \mathbf{p}) &= E_p\,.
\end{aligned}
\tag{34}
$$

Since purely magnetic fields do not produce pairs, we can restrict our attention for the Schwinger pair creation mechanism to electric fields. We assume for simplicity a spatially homogeneous but time-dependent electric field. The spatial homogeneity of the external fields and the initial condition (34) allow to reduce the constraint and transport equations to ordinary differential equations in time. The three-dimensional kinetic equations are obtained through the energy average as

$$
\begin{aligned}
\left(D_t^2 + 4E_p^2 + \frac{e}{3}\,(\partial_t\mathbf{E})\!\cdot\!\nabla_p\right)W_3(t, \mathbf{p}) &= 4\,\varepsilon(t, \mathbf{p})\,, \\
D_t\rho(t, \mathbf{p}) &= 0\,,
\end{aligned}
\tag{35}
$$

where

$$D_t = \partial_t + e\mathbf{E} \cdot \nabla_p \,. \tag{36}$$

To obtain a closed equation of motion for the Wigner function $W_3(t, \mathbf{p})$, we eliminate the energy density ε from the first equation of (35). To this end we multiply the transport equation (18) by p_0 from the left and then integrate it with respect to p_0:

$$D_t \varepsilon(t, \mathbf{p}) = \left(\frac{e}{12}(\partial_t \mathbf{E}) \cdot \nabla_p D_t + \frac{e}{12}(\partial_t^2 \mathbf{E}) \cdot \nabla_p + e\mathbf{p} \cdot \mathbf{E} \right) W_3(t, \mathbf{p}) \,. \tag{37}$$

We then apply the operator D_t on the first equation (35) and combine it with Eq. (37) to eliminate ε. We obtain

$$\left(D_t^3 + 4E_p^2 D_t + 4e\mathbf{E} \cdot \mathbf{p} \right) W_3(t, \mathbf{p}) = 0 \,, \tag{38}$$

which, together with the initial condition

$$W_3(t = -\infty) = \frac{1}{E_p} \,, \tag{39}$$

can be solved as an initial value problem for the Wigner function. We assume that the external electric field is switched on adiabatically in the far past and switched off adiabatically in the far future, $\mathbf{E}(t = -\infty) = \mathbf{E}(t = \infty) = 0$.

The momentum derivative hidden in the operator D_t complicates the solution of Eq. (38). We follow [14] and use the well-known method of characteristics in order to separate the momentum derivative and obtain an ordinary differential equation in time. One introduces a test Wigner function [14] ω_3 through

$$W_3(t, \mathbf{p}) = \int d^3\mathbf{p}_0 \, \omega_3(t, \mathbf{p}_0) \, \delta^{(3)}(\mathbf{p} - \mathbf{p}(t, \mathbf{p}_0)) \,, \tag{40}$$

where the function $\mathbf{p}(t, \mathbf{p}_0)$ is a solution of the classical equation of motion for a particle with initial momentum $\mathbf{p}_0$ in an external electric field:

$$\frac{d\mathbf{p}(t, \mathbf{p}_0)}{dt} = e\mathbf{E}(t) \,. \tag{41}$$

13

Its explicit form reads

$$\mathbf{p}(t, \mathbf{p}_0) = \mathbf{p}_0 + e \int_{-\infty}^{t} dt' \, \mathbf{E}(t') \,. \tag{42}$$

Substituting (40) into (38), the momentum derivative is absorbed into the classical motion, and the partial differential equation is converted into

$$\int d^3\mathbf{p}_0 \left[\left(\partial_t^3 + 4E_p^2 \partial_t + 4e\mathbf{E}\cdot\mathbf{p} \right) \omega_3(t, \mathbf{p}_0) \right] \delta^{(3)} \left(\mathbf{p} - \mathbf{p}(t, \mathbf{p}_0) \right) = 0 \,. \tag{43}$$

This is solved if $\omega_3(t, \mathbf{p}_0)$ satisfies the ordinary differential equation

$$\left(\partial_t^3 + 4E_p^2(t, \mathbf{p}_0) \, \partial_t + 4e\mathbf{E}\cdot\mathbf{p}(t, \mathbf{p}_0) \right) \omega_3(t, \mathbf{p}_0) = 0 \tag{44}$$

with the initial condition

$$\omega_3(t = -\infty, \mathbf{p}_0) = \frac{1}{E_p} \,, \tag{45}$$

where $\mathbf{p}(t, \mathbf{p}_0)$ satisfies Eq. (42) and $E_p(t, \mathbf{p}_0)$ is the correspondimg on-shell energy.

Please note the time dependence of the momentum $\mathbf{p}$ and the particle energy E_p arising from the classical equation of motion: when inserting the solution of (44) into (40) in order to construct the Wigner function W_3 from the test function ω_3, all the classical time dependence of W_3 resides in these functions $\mathbf{p}(t, \mathbf{p}_0)$, $E_p(t, \mathbf{p}_0)$, while the additional time dependence from quantum effects is described by the differential equation (44) for ω_3.

For later it will be useful to similarly introduce a test charge density $\varrho(t, \mathbf{p}_0)$, a test charge current $\mathcal{J}(t, \mathbf{p}_0)$, and a test energy density $\mathcal{E}(t, \mathbf{p}_0)$; for example,

$$\rho(t, \mathbf{p}) = \int d^3\mathbf{p}_0 \, \varrho(t, \mathbf{p}_0) \, \delta^{(3)} \left(\mathbf{p} - \mathbf{p}(t, \mathbf{p}_0) \right) \,. \tag{46}$$

They satisfy for arbitrary initial momentum $\mathbf{p}_0$ the following equations:

$$\begin{aligned} \partial_t \, \varrho(t, \mathbf{p}_0) &= 0 \,, \\ \partial_t \, \mathcal{E}(t, \mathbf{p}_0) &= \mathbf{E} \cdot \mathcal{J}(t, \mathbf{p}_0) \,, \end{aligned} \tag{47}$$

which follow from Eqs.(35) and (37), respectively. Their physical meaning is evident: The first equation shows that the net charge density always vanishes due to the homogeneous initial conditions (34). The second one is just Poynting's theorem of energy-momentum conservation.

2.3.2 Pair density

As pointed out in [21], the problem of pair creation in a homogeneous, but time-dependent electric field can be mapped onto the quantum mechanics problem of an oscillator with variable frequency. This will be exploited in the following treatment. We introduce an auxiliary function $\zeta(t)$ via

$$\omega_3(t) = |\zeta(t)|^2, \tag{48}$$

in terms of which the pair production problem (44,45)) is reduced to the solution of a quantum oscillator problem:

$$\partial_t^2 \zeta + E_p^2(t)\,\zeta = 0, \qquad |\zeta(t = -\infty)| = 1/\sqrt{E_p}. \tag{49}$$

Since we are interested in the total pair production yield at time $t \to \infty$, it is sufficient to study the asymptotic solutions of this equation. Due to its similarity with the time-dependent barrier-potential problem [22] in non-relativistic quantum mechanics, the asymptotic solutions can be easily written down in WKB approximation:

$$
\begin{aligned}
\zeta(t \to -\infty) &= e^{-iE_p^{-\infty}t}/\sqrt{E_p^{-\infty}}, \\
\zeta(t \to +\infty) &= C_1 e^{-iE_p^{\infty}t} + C_2 e^{iE_p^{\infty}t}.
\end{aligned}
\tag{50}
$$

Here we have already taken into account the initial condition for ζ, and $E_p^{-\infty}$ and E_p^{∞} are the asymptotic particle energies $E_p^{-\infty} = \sqrt{m^2 + \mathbf{p}_0^2}$ and $E_p^{\infty} = \sqrt{m^2 + (\mathbf{p}(t = \infty, \mathbf{p}_0))^2}$. The condition for applicability of the WKB approximation [23],

$$\partial_t E_p = \frac{e\mathbf{p}\cdot\mathbf{E}}{E_p} \ll E_p^2, \tag{51}$$

is satisfied in the limit $t \to \pm\infty$ since the electric field vanishes in this limit.

From these asymptotic solutions we see that the test Wigner function in the limit $t \to +\infty$ contains both oscillating and non-oscillating parts:

$$\omega_3(t \to \infty) = |\zeta(t \to \infty)|^2 = (|C_1|^2 + |C_2|^2) + (C_1 C_2^* e^{-2iE_p^{\infty}t} + \text{c.c.}). \tag{52}$$

The created pairs are separated and accelerated by the electric field in opposite directions, thereby generating a current. Writing this current as $\mathcal{J}(t) = e\mathbf{v}(\mathbf{t})n(t)$, where $n(t)$ is the total particle density (positive plus negative particles), and comparing with its expression in terms of the test Wigner function, namely, $\mathcal{J}(t) = e\mathbf{p}(t)\,\omega_3(t)$, we find

$$n(t) = E_p(t)\,\omega_3(t)\,. \tag{53}$$

Taking the time average of Eq. (52) which removes the rapidly oscillating parts, we thus find for the asymptotic value of the total particle density

$$n(t = \infty) = E_p^\infty\left(|C_1|^2 + |C_2|^2\right). \tag{54}$$

Due to the conservation law

$$\partial_t\left[(\partial_t\zeta)\zeta^* - (\partial_t\zeta^*)\zeta\right] = 0\,, \tag{55}$$

which is easily derived from (49), the expression in the square bracket is the same at $t = +\infty$ and $t = -\infty$ and can thus be evaluated with the initial conditions (50). We obtain

$$|C_1|^2 - |C_2|^2 = \frac{1}{E_p^\infty}\,. \tag{56}$$

Thus the oscillator is fully characterized by the ratio

$$r = \frac{|C_2|^2}{|C_1|^2}\,, \tag{57}$$

which can be interpreted (and calculated) as the transmission coefficient of the non-relativistic barrier-potential problem [22]. In terms of this ratio, the asymptotic particle density is given as

$$n(t = \infty) = \frac{1 + r}{1 - r} = 1 + 2\frac{r}{1 - r}\,. \tag{58}$$

The first term on the right-hand side is a vacuum contribution and stems from the Dirac sea of charged particles; it can not be measured and must be removed by renormalization [19]. The second term arises from the pair creation. The factor of 2 reminds us that both

particles and antiparticles contribute to the pair current. The asymptotic pair density in phase-space is thus finally obtained as

$$n_{pair}(t = \infty) = \frac{r}{1 - r} \,, \tag{59}$$

which is in full agreement with the results derived both from field theory [21] and from the transport theory in Feshbach-Villars representation [19].

Even though the final results of the two approaches to scalar transport theory, the equal-time and the energy averaging methods, are the same, we would like to point out one essential difference between these two procedures: The equal-time formulation has to rely on the Feshbach-Villars representation, since it requires field equations which contain only first-order time derivatives. As a result, the Wigner operator in the equal-time approach is a 2×2 matrix and first must be decomposed into its "spinor" components, in an analogous way as required for spinor QED which will be discussed in the following section. This matrix structure leads to a set of coupled kinetic equations for the "spinor" components of W_3. The energy averaging method works directly with the covariant Klein-Gordon equation. This is a scalar equation, and no such complication arises.

3 Spinor electrodynamics

3.1 Covariant version of the BGR equation

We now turn to the investigation of the transport theory for spin-$\frac{1}{2}$ particles interacting with an electromagnetic field. We start from the lagrangian density

$$\mathcal{L} = \bar{\psi}\Big(i\gamma^{\mu}(\partial_{\mu} + ieA_{\mu}) - m\Big)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \,, \tag{60}$$

which gives rise to the Dirac equations for the complex fields ψ and $\bar{\psi}$,

$$
\begin{aligned}
i\gamma^{\mu}(\partial_{\mu} + ieA_{\mu})\psi &= m\psi \,, \\
i(\partial_{\mu} - ieA_{\mu})\bar{\psi}\gamma^{\mu} &= -m\bar{\psi} \,.
\end{aligned}
\tag{61}
$$

The spinor Wigner operator is the four-dimensional Fourier transform of the gauge invariant density matrix

$$\Phi_4(x, y) = \psi\left(x + \tfrac{1}{2}y\right) \exp\left[ie \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, A(x + sy)\cdot y\right] \bar{\psi}\left(x - \tfrac{1}{2}y\right). \tag{62}$$

Unlike scalar electrodynamics, Φ_4 is now a 4×4 matrix in spin space, and the Wigner operator $\hat{W}$ is no longer self-adjoint. It behaves under hermitian conjugation like an ordinary γ-matrix,

$$\hat{W}_4^{\dagger}(x, p) = \gamma_0 \hat{W}_4(x, p)\gamma_0. \tag{63}$$

The evolution equation of the covariant Wigner operator again follows from the field equations of motion. Calculating the first-order derivatives of the density matrix and using the Dirac equations in a similar way as in the scalar case, we find a pair of equations for Φ_4 in terms of the operators D and Π,

$$\begin{aligned}
\left(\tfrac{1}{2}D_\mu(x, y) - i\Pi_\mu(x, y)\right)\gamma^0\gamma^\mu\Phi_4(x, y) &= -im\gamma^0\Phi_4(x, y), \\
\left(\tfrac{1}{2}D_\mu(x, y) + i\Pi_\mu(x, y)\right)\Phi_4(x, y)\gamma^\mu\gamma^0 &= im\Phi_4(x, y)\gamma^0.
\end{aligned} \tag{64}$$

The γ_0-matrices have been included in order to facilitate comparison with the three-dimensional equal-time approach of BGR [14].

Multiplying (64) by another γ_0-matrix from the right and taking the Wigner transform, we derive the following kinetic equations for the Wigner operator:

$$\left(\tfrac{1}{2}\hbar D_\mu(x, p) - i\Pi_\mu(x, p)\right)\gamma^0\gamma^\mu\hat{W}_4(x, p)\gamma^0 = -im\gamma^0\hat{W}_4(x, p)\gamma^0, \tag{65}$$

$$\left(\tfrac{1}{2}\hbar D_\mu(x, p) + i\Pi_\mu(x, p)\right)\hat{W}_4(x, p)\gamma^\mu = im\hat{W}_4(x, p). \tag{66}$$

With an eye on the semiclassical expansion below, we have again displayed the $\hbar$-dependence explicitly. We note that these two equations of motion in phase space, like the two Dirac equations (61) in coordinate space, are adjoints of each other. Therefore, either one of the Eqs. (65,66) provides a complete description of the Wigner operator. By adding and

subtracting these two equations, respectively, we can get two different, now self-adjoint equations:

$$\tfrac{1}{2}\hbar D_\mu\{\gamma^0\gamma^\mu,\ \hat{W}_4\gamma^0\} - i\Pi_\mu[\gamma^0\gamma^\mu,\ \hat{W}_4\gamma^0] \;=\; -im[\gamma^0,\ \hat{W}_4\gamma^0]\,, \tag{67}$$

$$\tfrac{1}{2}\hbar D_\mu[\gamma^0\gamma^\mu,\ \hat{W}_4\gamma^0] - i\Pi_\mu\{\gamma^0\gamma^\mu,\ \hat{W}_4\gamma^0\} \;=\; -im\{\gamma,\ \hat{W}_4\gamma^0\}\,. \tag{68}$$

Obviously (67) and (68) are symmetric under an exchange of commutators and anticommutators. Their self-adjoint properties are easily proven with the aid of Eq. (63). Since these two self-adjoint equations are equivalent to either of the equations (65,66), we have now three choices for a full description of the Wigner operator: Eq. (65), Eq.(66), or Eq.(67) together with Eq.(68). Eqs.(67) and (68) do not separately provide a complete description.

We would like to note another important feature of Eq.(67): The commutator in the second term of the left-hand side leads to the automatic disappearance of p_0 contained in the operator $\Pi_0(x,p)$. Since the only remaining p_0-dependence is in $\hat{W}_4$, this renders the energy average very simple. The result of an integration over p_0 is just the BGR equation from the equal-time formulation,

$$\hbar D_t\hat{W}_3(x,\mathbf{p}) = -\tfrac{1}{2}\hbar\mathbf{D}{\cdot}\{\gamma^0\boldsymbol{\gamma},\ \hat{W}_3(x,\mathbf{p})\} - i\boldsymbol{\Pi}{\cdot}[\gamma^0\boldsymbol{\gamma},\ \hat{W}_3(x,\mathbf{p})] - im[\gamma^0,\ \hat{W}_3(x,\mathbf{p})]\,, \tag{69}$$

with

$$
\begin{aligned}
D_t(x,\mathbf{p}) &= \partial_t + e\int_{-\frac{1}{2}}^{\frac{1}{2}} ds\,\mathbf{E}(\mathbf{x}+is\hbar\nabla_p,t){\cdot}\nabla_p\,, \\[2mm]
\mathbf{D}(x,\mathbf{p}) &= \nabla + e\int_{-\frac{1}{2}}^{\frac{1}{2}} ds\,\mathbf{B}(\mathbf{x}+is\hbar\nabla_p,t)\times\nabla_p\,, \\[2mm]
\boldsymbol{\Pi}(x,\mathbf{p}) &= \mathbf{p} - ie\hbar\int_{-\frac{1}{2}}^{\frac{1}{2}} ds\,s\,\mathbf{B}(\mathbf{x}+is\hbar\nabla_p,t)\times\nabla_p\,,
\end{aligned} \tag{70}
$$

where we have employed the BGR definition of the equal-time Wigner operator [14]:

$$
\begin{aligned}
\hat{W}_3(x,\mathbf{p}) &= \int dp_0\,\hat{W}_4(x,p)\,\gamma_0 \\[2mm]
&= \int d^3\mathbf{y}\,e^{-i\mathbf{p}\cdot\mathbf{y}}\,\psi\left(\mathbf{x}+\tfrac{1}{2}\mathbf{y},t\right)\exp\left[-ie\int_{-\frac{1}{2}}^{\frac{1}{2}} ds\,\mathbf{A}(\mathbf{x}+s\mathbf{y},t)\cdot\mathbf{p}\right]\psi^\dagger\left(\mathbf{x}-\tfrac{1}{2}\mathbf{y},t\right).
\end{aligned} \tag{71}
$$

Clearly equation (67) is the covariant version of the BGR equation. However, as we have stressed, (67) is not complete. Only together with the self-adjoint equation (68) one obtains a full description for the Wigner operator. Therefore the covariant version of the BGR equation is only one part of the full covariant transport theory. The question thus arises whether on the three-dimensional level the BGR equation can be complete or not. To answer this question we will reduce Eq. (68) to the three-dimensional level by performing an energy average, too, and study its implications.

3.2   Energy average of full covariant theory

The two self-adjoint equations (67) and (68) have a complicated structure due to the occurrence of commutators and anticommutators. But since they are equivalent to either equation (65) or (66), we can consider equation (65) instead. To simplify the calculation further, we even remove the γ_0-matrices and thus obtain the VGE [13] equation:

$$\left[\gamma^\mu \left(\Pi_\mu + \tfrac{1}{2}i\hbar D_\mu\right) - m\right] \hat{W}_4(x,p) = 0 \,. \tag{72}$$

As in the scalar case, we will from now on consider the electromagnetic field as classical and pass from the Wigner operator to the Wigner function by leaving off the hats.

The Wigner function in spinor electrodynamics is a complex 4×4 matrix. VGE discussed its spinor decomposition and derived a set of coupled equations for the components of W_4. Because of their characteristic transformation properties under Lorentz transformations, it is convenient to choose the 16 matrices $\Gamma_i = \{1,\ i\gamma_5,\ \gamma_\mu,\ \gamma_\mu\gamma_5,\ \tfrac{1}{2}\sigma_{\mu\nu}\}$ as the basis for an expansion of the Wigner function in spin space:

$$W_4(x,p) = \tfrac{1}{4}\left[F(x,p) + i\gamma_5 P(x,p) + \gamma_\mu V^\mu(x,p) + \gamma_\mu\gamma_5 A^\mu(x,p) + \tfrac{1}{2}\sigma_{\mu\nu}S^{\mu\nu}(x,p)\right]\,. \tag{73}$$

All the components F, P, V_μ, A_μ, and $S_{\mu\nu}$ are real functions since the basis elements Γ_i transform under hermitian conjugation like W_4 itself, $\Gamma_i^\dagger = \gamma_0\Gamma_i\gamma_0$ (see Eq. (63)). They can thus be interpreted as physical phase-space densities. The expansion (73) decomposes the VGE equation into 5 coupled Lorentz covariant equations for the spinor components. Since these components are real and the operators D and Π are self-adjoint, one can

separate the real and imaginary parts of these 5 complex equations to obtain 10 real equations:

$$\Pi^\mu V_\mu = mF \,,$$

$$\hbar D^\mu A_\mu = 2mP \,,$$

$$\Pi_\mu F - \tfrac{1}{2}\hbar D^\nu S_{\nu\mu} = mV_\mu \,,$$

$$-\hbar D_\mu P + \epsilon_{\mu\nu\sigma\rho}\Pi^\nu S^{\sigma\rho} = 2mA_\mu \,,$$

$$\tfrac{1}{2}\hbar(D_\mu V_\nu - D_\nu V_\mu) + \epsilon_{\mu\nu\sigma\rho}\Pi^\sigma A^\rho = mS_{\mu\nu} \,, \tag{74}$$

and

$$\hbar D^\mu V_\mu = 0 \,,$$

$$\Pi^\mu A_\mu = 0 \,,$$

$$\tfrac{1}{2}\hbar D_\mu F = -\Pi^\nu S_{\nu\mu} \,,$$

$$\Pi_\mu P = -\tfrac{1}{4}\hbar\epsilon_{\mu\nu\sigma\rho}D^\nu S^{\sigma\rho} \,,$$

$$\Pi_\mu V_\nu - \Pi_\nu V_\mu = \tfrac{1}{2}\hbar\epsilon_{\mu\nu\sigma\rho}D^\sigma A^\rho \,. \tag{75}$$

Peforming an energy average of the expansion of W_4, (73), we obtain a full mapping from the four- to the three-dimensional components introduced in [14],

$$W_3(x,\mathbf{p}) = \tfrac{1}{4}\Big[f_0(x,\mathbf{p}) + \gamma_5 f_1(x,\mathbf{p}) - i\gamma_0\gamma_5 f_2(x,\mathbf{p}) + \gamma_0 f_3(x,\mathbf{p}) \tag{76}$$

$$+\gamma_5\boldsymbol{\gamma}{\cdot}\mathbf{g}_0(x,\mathbf{p}) + \gamma_0\boldsymbol{\gamma}{\cdot}\mathbf{g}_1(x,\mathbf{p}) - i\boldsymbol{\gamma}{\cdot}\mathbf{g}_2(x,\mathbf{p}) - \gamma_5\boldsymbol{\gamma}{\cdot}\mathbf{g}_3(x,\mathbf{p})\Big] \,,$$

namely,

$$f_0(x,\mathbf{p}) = \int dp_0\, V_0(x,p) \,,$$

$$f_1(x,\mathbf{p}) = -\int dp_0\, A_0(x,p) \,,$$

$$f_2(x,\mathbf{p}) = \int dp_0\, P(x,p) \,,$$

$$f_3(x,\mathbf{p}) = \int dp_0\, F(x,p) \,,$$

$$\mathbf{g}_0(x,\mathbf{p}) = -\int dp_0\, \mathbf{A}(x,p) \,,$$

$$\mathbf{g}_1(x,\mathbf{p}) = \int dp_0\, \mathbf{V}(x,p) \,,$$

$$g_2^i(x,\mathbf{p}) = -\int dp_0\, S^{0i}(x,p) \,, \quad i = 1,2,3 \,,$$

$$g_3^i(x,\mathbf{p}) = \tfrac{1}{2}\epsilon^{ijk}\int dp_0\, S_{jk}(x,p) \,, \quad i = 1,2,3 \,. \tag{77}$$

As shown in [14, 20], the physically interesting densities such as charge current, energy momentum tensor and angular momentum tensor, can also be expressed in terms of these spinor components. For example, f_0 and $\mathbf{g}_0$ are the charge density and the spin density, respectively.

The VGE equations (74,75) can be divided into two groups. Group I contains the operator $\Pi_0(x, p)$ which involves p_0; group II contains no p_0-dependence except for the one in the Wigner function itself. For group I the energy average is straightforward. The result is in complete agreement with the spinor decomposition of the BGR equation [14]:

$$\hbar(D_t f_0 + \mathbf{D}\cdot\mathbf{g}_1) = 0\,,$$

$$\hbar(D_t f_1 + \mathbf{D}\cdot\mathbf{g}_0) = -2m f_2\,,$$

$$\hbar D_t f_2 + 2\mathbf{\Pi}\cdot\mathbf{g}_3 = 2m f_1\,,$$

$$\hbar D_t f_3 - 2\mathbf{\Pi}\cdot\mathbf{g}_2) = 0\,,$$

$$\hbar(D_t\mathbf{g}_0 + \mathbf{D}f_1) - 2\mathbf{\Pi}\times\mathbf{g}_1 = 0\,,$$

$$\hbar(D_t\mathbf{g}_1 + \mathbf{D}f_0) - 2\mathbf{\Pi}\times\mathbf{g}_0 = -2m\mathbf{g}_2\,,$$

$$\hbar(D_t\mathbf{g}_2 + \mathbf{D}\times\mathbf{g}_3) + 2\mathbf{\Pi}f_3 = 2m\mathbf{g}_1\,,$$

$$\hbar(D_t\mathbf{g}_3 - \mathbf{D}\times\mathbf{g}_2) - 2\mathbf{\Pi}f_2 = 0\,. \tag{78}$$

Due to the additional p_0-dependence from the operator Π_0, the equations in group II can not be completely reduced to expressions from the set of three-dimensional components given in Eq. (77). They contain additionally higher p_0-moments of the four-dimensional components,

$$\int dp_0\, p_0\, V_0 - \mathbf{\Pi}\cdot\mathbf{g}_1 + \tilde{\Pi}_0 f_0 = m f_3\,,$$

$$\int dp_0\, p_0\, A_0 + \mathbf{\Pi}\cdot\mathbf{g}_0 - \tilde{\Pi}_0 f_1 = 0\,,$$

$$\int dp_0\, p_0\, P + \tfrac{1}{2}\hbar\mathbf{D}\cdot\mathbf{g}_3 + \tilde{\Pi}_0 f_2 = 0\,,$$

$$\int dp_0\, p_0\, F - \tfrac{1}{2}\hbar\mathbf{D}\cdot\mathbf{g}_2 + \tilde{\Pi}_0 f_3 = m f_0\,,$$

$$\int dp_0\, p_0\, \mathbf{A} + \tfrac{1}{2}\hbar\mathbf{D}\times\mathbf{g}_1 + \mathbf{\Pi}f_1 - \tilde{\Pi}_0\mathbf{g}_0 = -m\mathbf{g}_3\,,$$

$$\int dp_0\, p_0\, \mathbf{V} - \tfrac{1}{2}\hbar\mathbf{D}\times\mathbf{g}_0 - \mathbf{\Pi}f_0 + \tilde{\Pi}_0\mathbf{g}_1 = 0\,,$$

$$\int dp_0\, p_0\, S^{0i}\mathbf{e}_i - \tfrac{1}{2}\hbar\mathbf{D}f_3 + \mathbf{\Pi}\times\mathbf{g}_3 - \tilde{\Pi}_0\mathbf{g}_2 = 0\,,$$

$$\int dp_0\, p_0\, S_{jk}\epsilon^{jki}\mathbf{e}_i - \hbar\mathbf{D}f_2 + 2\mathbf{\Pi}\times\mathbf{g}_2 + 2\tilde{\Pi}_0\mathbf{g}_3 = 2m\mathbf{g}_0\,, \tag{79}$$

where we introduced the three-dimensional operator

$$\tilde{\Pi}_0(x, \mathbf{p}) = ie\hbar \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, s\, \mathbf{E}(\mathbf{x} + is\hbar\nabla_p, t)\cdot\nabla_p\,. \tag{80}$$

This second set of coupled equations for the components of W_3 does not include the operator D_t. It corresponds to the energy average of the spinor decomposition of the second self-adjoint equation (68), where D_t drops out due to the commutator in the first term. These equations are therefore constraint equations rather than equations of motion. They arise in addition to the BGR equations.

3.3 Classical limit

Before discussing the non-perturbative consequences of Eqs. (79) we show how they constrain the transport equations in the classical limit. As $\hbar \to 0$, the original VGE equation can be written in the following quadratic form:

$$(p^2 - m^2)\, W_4(x, p) = 0\,. \tag{81}$$

This shows that the classical Wigner operator and hence all its spinor components are all on the mass shell, and thus they have positive and negative energy solutions similar to Eq.(23),

$$f_i^{+(0)}(x, \mathbf{p})\, \delta(p_0 - E_p), \quad f_i^{-(0)}(x, \mathbf{p})\, \delta(p_0 + E_p)\,,$$

$$\mathbf{g}_i^{+(0)}(x, \mathbf{p})\, \delta(p_0 - E_p), \quad \mathbf{g}_i^{-(0)}(x, \mathbf{p})\, \delta(p_0 + E_p)\,,$$

$$i = 0, 1, 2, 3\,. \tag{82}$$

With this the remaining p_0-integrals in Eqs. (79) can be done, and everything can be fully expressed in terms of the three-dimensional functions (77). Eqs. (79) constribute

altogether six independent constraints for the positive and negative energy parts of the spinor components:

$$f_1^{\pm(0)} = \pm \frac{\mathbf{p} \cdot \mathbf{g}_0^{\pm(0)}}{E_p},$$

$$f_2^{\pm(0)} = 0,$$

$$f_3^{\pm(0)} = \pm \frac{m}{E_p} f_0^{\pm(0)},$$

$$\mathbf{g}_1^{\pm(0)} = \pm \frac{\mathbf{p}}{E_p} f_0^{\pm(0)},$$

$$\mathbf{g}_2^{\pm(0)} = \frac{\mathbf{p} \times \mathbf{g}_0^{\pm(0)}}{m},$$

$$\mathbf{g}_3^{\pm(0)} = \pm \frac{E_p^2 \mathbf{g}_0^{\pm(0)} - (\mathbf{p} \cdot \mathbf{g}_0^{\pm(0)})\mathbf{p}}{mE_p}. \tag{83}$$

In addition, the classical limit [20] of the BGR equations in (78) gives another four constraints. Three of them are, however, already included in (83), namely, the equations for $f_2^{+(0)}, \mathbf{g}_1^{+(0)}$ and $\mathbf{g}_2^{+(0)}$. The last one, $f_1^{+(0)} = \mathbf{p} \cdot \mathbf{g}_3^{+(0)}/m$, is not independent of the above equations either, but can be obtained through a linear combination of Eqs. (83). Therefore, Eqs. (83) provide a complete set of constraint equations for the three-dimensional spinor components in the classical limit. There remain only two independent components, for example, the charge density $f_0^{(0)}$ and the spin density $\mathbf{g}_0^{(0)}$. The classical limit of the BGR equations yields only the positive energy parts of the latter 4 constraints, and misses the first and the third additional equations from (83).

The classical transport equations for $f_0^{(0)}$ and $\mathbf{g}_0^{(0)}$ arise from the first order in $\hbar$ of the kinetic equations. From the BGR equations in (78), we have

$$(\partial_t + e\mathbf{E} \cdot \nabla_p) f_0^{\pm(0)} + (\nabla + e\mathbf{B} \times \nabla_p) \cdot \mathbf{g}_1^{\pm(0)} = 0,$$

$$(\partial_t + e\mathbf{E} \cdot \nabla_p) \mathbf{g}_3^{\pm(0)} - (\nabla + e\mathbf{B} \times \nabla_p) \times \mathbf{g}_2^{\pm(0)}$$

$$+ \frac{\mathbf{p}}{m} \Big((\partial_t + e\mathbf{E} \cdot \nabla_p) f_1^{\pm(0)} + (\nabla + e\mathbf{B} \times \nabla_p) \cdot \mathbf{g}_0^{\pm(0)} \Big) = 0. \tag{84}$$

Using the constraint equations, some straightforward manipulations lead to the following decoupled Vlasov-type equation for the charge density,

$$\partial_t f_0^{\pm(0)} \pm \mathbf{v} \cdot \nabla f_0^{\pm(0)} + e(\mathbf{E} \pm \mathbf{v} \times \mathbf{B}) \cdot \nabla_p f_0^{\pm(0)} = 0, \tag{85}$$

and the three-dimensional kinetic equation for the spin density,

$$\partial_t \mathbf{g}_0^{\pm(0)} \pm (\mathbf{v}\cdot\nabla)\mathbf{g}_0^{\pm(0)} + e\big[(\mathbf{E}\pm\mathbf{v}\times\mathbf{B})\cdot\nabla_p\big]\mathbf{g}_0^{\pm(0)}$$
$$-\frac{e}{E_p^2}\big[(\mathbf{p}\cdot\mathbf{g}_0^{\pm(0)})\mathbf{E} - (\mathbf{E}\cdot\mathbf{p})\mathbf{g}_0^{\pm(0)}\big] \pm \frac{e}{E_p}\mathbf{B}\times\mathbf{g}_0^{\pm(0)} = 0\,. \tag{86}$$

Through the energy averaging method, it is easy to prove that the above equation is just the three-dimensional formulation of the covariant transport equation for the classical axial vector $A_\mu^{(0)}$,

$$p^\mu(\partial_\mu - eF_{\mu\nu}\partial_p^\nu)A_\sigma^{(0)} = eF_{\sigma\rho}A^{\rho(0)}\,, \tag{87}$$

which can be derived from the first order in $\hbar$ of the linear equations (74) and (75). From the discussions in [13], this transport equation for $A_\mu^{(0)}$ is equivalent to the covariant Bargmann-Michel-Telegdi (BMT) equation [5, 13, 24] for a spinning particle in a constant external field,

$$m\frac{ds^\mu}{d\tau} = eF^{\mu\nu}(\tau)s_\nu(\tau)\,, \tag{88}$$

where $s_\mu = A_\mu^{(0)}/(A^{(0)}\cdot A^{(0)})^{1/2}$ is the covariant spin phase-space density. Therefore, the kinetic equation (86) for the spin density $\mathbf{g}_0^{\pm(0)}$ can be recognized as the three-dimensional BMT equation. It describes the procession of the spin-polarization in a homogeneous external electromagnetic field.

Equations (85) and (86) can be further simplified by introducing classical particle and antiparticle distribution functions in an analogous way to the case of scalar QED:

$$
\begin{aligned}
f_i(x,\mathbf{p}) &= f_i^{+(0)}(x,\mathbf{p})\,, \\
\mathbf{g}_i(x,\mathbf{p}) &= \mathbf{g}_i^{+(0)}(x,\mathbf{p})\,, \\
\bar{f}_i(x,\mathbf{p}) &= f_i^{-(0)}(x,-\mathbf{p})\,, \\
\bar{\mathbf{g}}_i(x,\mathbf{p}) &= \mathbf{g}_i^{-(0)}(x,-\mathbf{p})\,.
\end{aligned}
\tag{89}
$$

The resulting equations for the particle distributions read

$$\partial_t f_0 + \mathbf{v}\cdot\nabla f_0 + e(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_p f_0 = 0\,, \tag{90}$$

$$\partial_t \mathbf{g}_0 + (\mathbf{v}\cdot\nabla)\mathbf{g}_0 + e\Big[(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_p\Big]\mathbf{g}_0 - \frac{e}{E_p^2}\Big[(\mathbf{p}\cdot\mathbf{g}_0)\mathbf{E} - (\mathbf{E}\cdot\mathbf{p})\mathbf{g}_0\Big] + \frac{e}{E_p}\mathbf{B}\times\mathbf{g}_0 = 0 \,; \quad (91)$$

those for the antiparticle distributions $\bar{f}_0$ and $\bar{\mathbf{g}}_0$ differ only by a minus sign in front of the electric charge e.

The other particle and antiparticle distribution functions can be derived from the constraints (83):

$$f_1 = \frac{\mathbf{p}\cdot\mathbf{g}_0}{E_p}, \qquad \bar{f}_1 = \frac{\mathbf{p}\cdot\bar{\mathbf{g}}_0}{E_p},$$

$$f_2 = \bar{f}_2 = 0,$$

$$f_3 = \frac{m}{E_p}f_0, \qquad \bar{f}_3 = -\frac{m}{E_p}\bar{f}_0,$$

$$\mathbf{g}_1 = \frac{\mathbf{p}}{E_p}f_0, \qquad \bar{\mathbf{g}}_1 = \frac{\mathbf{p}}{E_p}\bar{f}_0,$$

$$\mathbf{g}_2 = \frac{\mathbf{p}\times\mathbf{g}_0}{m}, \qquad \bar{\mathbf{g}}_2 = -\frac{\mathbf{p}\times\bar{\mathbf{g}}_0}{m},$$

$$\mathbf{g}_3 = \frac{E_p^2\mathbf{g}_0 - (\mathbf{p}\cdot\mathbf{g}_0)\mathbf{p}}{E_p m}, \qquad \bar{\mathbf{g}}_3 = -\frac{E_p^2\bar{\mathbf{g}}_0 - (\mathbf{p}\cdot\bar{\mathbf{g}}_0)\mathbf{p}}{E_p m}. \quad (92)$$

The discussion above demonstrates the incompleteness of the equal-time BGR formulation, (69) and (78), in the classical limit. Without the additional constraints (79) one has [20] 4 independent components, $f_0^{(0)}, f_3^{(0)}, \mathbf{g}_0^{(0)}$ and $\mathbf{g}_3^{(0)}$, and one obtains two groups of coupled equations, where the first one couples $f_0^{(0)}$ to $f_3^{(0)}$ and the second one couples $\mathbf{g}_0^{(0)}$ to $\mathbf{g}_3^{(0)}$. The additional constraints arising from the complete covariant formulation allow to decouple $f_0^{(0)}$ from $f_3^{(0)}$ and $\mathbf{g}_0^{(0)}$ from $\mathbf{g}_3^{(0)}$. Therefore, we need only one scalar and one vector component to completely describe the classical behavior of the Wigner operator. In fact, already VGE [13] pointed out in their covariant formulation that to any finite order in $\hbar$ the pseudoscalar P, vector V^μ and antisymmetric tensor $S^{\mu\nu}$ components can be expressed in terms of the scalar F and axial vector A^μ components. Furthermore, the general relationship $\Pi_\mu A^\mu = 0$ in (74) shows that A_0 is not an independent component either. Thus only 4 of the 16 components of the Wigner function are dynamically independent. Through our mapping between the four- and three-dimensional components, which results from taking the energy average, we are thus able to construct transport equations with only 2 independent densities, the scalar mass density f_3 and the vector spin density $\mathbf{g}_0$, to any order of the semiclassical expansion.

3.4   General kinetic equations in three-dimensional form

In this subsection, we go beyond the semiclassical expansion and study the full quantum kinetic equations in three-dimensional form. This is important for the treatment of non-perturbative processes like pair creation in strong electric fields. Beyond the classical limit, the classical mass-shell condition is generally violated. The quantum Wigner function is no longer a δ-function in p_0 located at the mass-shell energy, and for the elimination of the higher p_0-moments of the covariant spinor components from Eqs. (79) we must follow a different path. Following the procedure from Section 2.3.1, we multiply the equations in group I of (74,75) by p_0 from the left and then take the energy average of these equations. We find

$$
D_t \int dp_0\, p_0\, V_0 + \mathbf{D}\cdot \int dp_0\, p_0\, \mathbf{V} + I f_0 + \mathbf{J}\cdot\mathbf{g}_1 = 0 \,,
$$

$$
D_t \int dp_0\, p_0\, A_0 + \mathbf{D}\cdot \int dp_0\, p_0\, \mathbf{A} - I f_1 - \mathbf{J}\cdot\mathbf{g}_0 = 2m \int dp_0\, p_0\, P \,,
$$

$$
D_t \int dp_0\, p_0\, P + I f_2 + 2\,\mathbf{K}\cdot\mathbf{g}_3 = -\mathbf{\Pi}\cdot \int dp_0\, p_0\, \epsilon^{ijk} S_{jk}\mathbf{e}_i - 2m \int dp_0\, p_0\, A_0 \,,
$$

$$
\tfrac{1}{2}\Big[D_t \int dp_0\, p_0\, F + I f_3 - 2\,\mathbf{K}\cdot\mathbf{g}_2\Big] = -\mathbf{\Pi}\cdot \int dp_0\, p_0\, S^{0i}\mathbf{e}_i \,,
$$

$$
D_t \int dp_0\, p_0\, \mathbf{A} + \mathbf{D} \int dp_0\, p_0\, A_0 - I\mathbf{g}_0 - \mathbf{J} f_1 + 2\,\mathbf{K}\times\mathbf{g}_1 = 2\,\mathbf{\Pi} \times \int dp_0\, p_0\, \mathbf{V} \,,
$$

$$
\tfrac{1}{2}\Big[D_t \int dp_0\, p_0\, \mathbf{V} + \mathbf{D} \int dp_0\, p_0\, V_0 + I\mathbf{g}_1 + \mathbf{J} f_0 - 2\,\mathbf{K}\times\mathbf{g}_0\Big] =
$$
$$
-\mathbf{\Pi}\times \int dp_0\, p_0\, \mathbf{A} + m \int dp_0\, p_0\, S^{0i}\mathbf{e}_i \,,
$$

$$
D_t \int dp_0\, p_0\, S^{0i}\mathbf{e}_i - \frac{1}{2}\mathbf{D}\times \int dp_0\, p_0\, \epsilon^{ijk} S_{jk}\mathbf{e}_i - I\mathbf{g}_2 - \mathbf{J}\times\mathbf{g}_3 - 2\,\mathbf{K} f_3 =
$$
$$
2\,\mathbf{\Pi} \int dp_0\, p_0\, F - 2m \int dp_0\, p_0\, \mathbf{V} \,,
$$

$$
\tfrac{1}{2}\Big[\tfrac{1}{2}D_t \int dp_0\, p_0\, \epsilon^{ijk} S_{jk}\mathbf{e}_i + \mathbf{D}\times \int dp_0\, p_0\, S^{0i}\mathbf{e}_i + I\mathbf{g}_3 - \mathbf{J}\times\mathbf{g}_2 - 2\,\mathbf{K} f_2\Big] =
$$
$$
\mathbf{\Pi} \int dp_0\, p_0\, P \,, \tag{93}
$$

with

$$
I(x,\mathbf{p}) = ie \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, s\, (\partial_t \mathbf{E})(\mathbf{x}+is\nabla_p, t)\cdot\nabla_p \,,
$$

$$
\mathbf{J}(x,\mathbf{p}) = ie \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, s\, (\partial_t \mathbf{B})(\mathbf{x}+is\nabla_p, t) \times \nabla_p - e \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, \mathbf{E}(\mathbf{x}+is\nabla_p, t) \,,
$$

$$
\mathbf{K}(x,\mathbf{p}) = e \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, s^2\, (\partial_t \mathbf{B})(\mathbf{x}+is\nabla_p, t) \times \nabla_p + ie \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, s\, \mathbf{E}(\mathbf{x}+is\nabla_p, t) \,. \tag{94}
$$

We now combine Eqs. (79) and (93) and eliminate all higher p_0-moments. In order to arrive at a set of independent contraints we also use the BGR equations and remove all information already contained in them. After a straightforward but tedious calculation we finally derive the following constraints:

$$Lf_0 + \mathbf{M} \cdot \mathbf{g}_1 = 0\,,$$

$$Lf_1 + \mathbf{M} \cdot \mathbf{g}_0 = 0\,,$$

$$Lf_2 + 2\mathbf{N} \cdot \mathbf{g}_3 = 0\,,$$

$$Lf_3 - 2\mathbf{N} \cdot \mathbf{g}_2 = 0\,,$$

$$L\mathbf{g}_0 - \mathbf{M}f_1 - 2\mathbf{N} \times \mathbf{g}_1 = 0\,,$$

$$L\mathbf{g}_1 + \mathbf{M}f_0 + 2\mathbf{N} \times \mathbf{g}_0 = 0\,,$$

$$L\mathbf{g}_2 + \mathbf{M} \times \mathbf{g}_3 - 2\mathbf{N}f_3 = 0\,,$$

$$L\mathbf{g}_3 - \mathbf{M} \times \mathbf{g}_2 + 2\mathbf{N}f_2 = 0\,, \tag{95}$$

with

$$L(x,\mathbf{p}) = ie \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, s \left(\nabla \times \mathbf{B}(\mathbf{x} + is\nabla_p, t) \right) \cdot \nabla_p\,,$$

$$\mathbf{M}(x,\mathbf{p}) = ie \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, s\, \nabla \Big(\mathbf{E}(\mathbf{x} + is\nabla_p, t) \cdot \nabla_p \Big) + e \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \left(\mathbf{E}(\mathbf{x} + is\nabla_p, t) - \mathbf{E}(x) \right),$$

$$\mathbf{N}(x,\mathbf{p}) = \tfrac{1}{4} e \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, (\nabla_p \cdot \nabla)\, \mathbf{E}(\mathbf{x} + is\nabla_p, t) + \mathbf{K}(x,\mathbf{p})\,. \tag{96}$$

The complete set of kinetic equations for the Wigner function $W_3(x,\mathbf{p})$ (resp. its spinor components) is given by the BGR equations (78) and the constraints (95). These equations together form the complete transport theory in its three-dimensional formulation. They are fully equivalent to the covariant kinetic approach defined by either of the two equations (65) or by the pair of self-adjoint equations (67,68). While the equations in (95) do not involve time derivatives and thus do not by themselves describe transport, they provide essential constraints for the Wigner operator.

The constraints (95) hold for arbitrary external fields. For homogeneous fields, the spatial derivatives of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ in the operators $L, \mathbf{M}$ and $\mathbf{N}$ vanish, and the constraints (95) disappear identically. This shows that the equal-time BGR theory is in fact complete for the case of a spatially constant external field, i.e. for the case studied in [14]. The application of the BGR theory in [14] to the problem of pair creation in spatially homogeneous but time-dependent electric fields is therefore safe.

4 Conclusions

Relativistic kinetic theory, formulated in terms of gauge covariant Wigner operators, provides a useful method to study transport problems in quantum theory, especially the phase-space evolution of high temperature and density plasmas. There exist two paths to formulate such a transport theory, starting from 4-dimensional or from 3-dimensional momentum space, respectively. In this paper we have shown a way to connect these two approaches directly and arrived at the 3-dimensional formulation by taking an energy average of the covariant 4-dimensional formulation.

We demonstrated the method explicitly for scalar and spinor QED with external electromagnetic fields. In scalar QED, our approach started directly from the Klein-Gordon equation and thus avoided the complications connected with the 2×2 matrix structure of the Feshbach-Villars representation [15]. Using the energy averaging method and taking the semiclassical limit, we directly arrived at the classical Vlasov equation for on-shell particles. We pointed out that there are no first-order quantum corrections due to the absence of spin. This may partially explain the success of the classical Vlasov approach as a good approximation for the description of many quantum systems involving scalar fields [25].

Beyond the semiclassical expansion, we could still perform the energy average analytically for the case of spatially constant electromagnetic fields, and we recovered the correct result for the pair creation rate in a homogeneous electric field. Our calculation was much easier than the one based on 3-dimensional transport theory in the Feshbach-Villars representation, due to the Lorentz scalar nature of the Wigner function employed by us.

In spinor electrodynamics, we concentrated on the question of completeness of the 3-dimensional (BGR) transport theory. Employing the energy averaging method, we identified a covariant version of the equal-time BGR equation, but found that this covariant equation is not equivalent to the original Dirac equation. Integrating instead the covariant VGE equation, which forms a complete description for the spinor Wigner

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function, over the energy p_0, we derived a set of 3-dimensional (equal-time) transport and constraint equations which is complete and contains the BGR equations as a subset. The additional equations can be understood as resulting from the energy average of the 4-dimensional constraint or generalized mass-shell condition. They cannot be obtained by the direct equal-time approach of BGR. In the semiclassical limit they allow to reduce the number of independent spinor components of the Wigner function from 8 in the BGR approach to 4, one scalar charge density and 3 vector components of the spin density. The equations for the charge density and for the spin density decouple, and each of them satisfies a Vlasov-type transport equation in (6+1)-dimensional phase space. We also derived the additional constraints explicitly for the full quantum case and showed that they vanish only for homogeneous external electromagnetic fields.

Either of the two formulations has its advantages and disadvantages. The approach based on 4-dimensional momentum space (8-dimensional phase space) is manifestly Lorentz covariant, but setting it up as an initial value problem poses certain difficulties [14]: since the covariant Wigner function is a 4-dimensional Wigner transform of the density matrix, its calculation at $t = -\infty$ requires knowledge of the fields at all times. Since this knowledge does not a priori exist, the covariant transport equations can only be solved with phenomenologically motivated forms for the Wigner function in the far past. The formulation in 3-dimensional momentum space (i.e. (6+1)-dimensional phase space) does not have this problem: it requires the density matrix only at equal times, and the Wigner function at $t = -\infty$ can be directly calculated from the fields at $t = -\infty$. Setting up the initial value problem is therefore straightforward in this approach. For the calculation of the pair creation rate in homogeneous electric fields this seems to be crucial: also our calculation was based on the 3-dimensional version after taking the energy average.

In both formulations the transport equations are supplemented by constraint equations. In the covariant approach this is essentially one 4×4 matrix equation, and it is easily interpreted as a generalized mass-shell condition. In the equal-time approach the constraints are less transparent, and their derivation via the energy averaging procedure

is actually rather tedious. For homogeneous external fields, however, they disappear to all orders in $\hbar$, so for this particular case the equal-time approach has a decisive advantage over the covariant approach where this does not happen. In all other cases a large number of constraints, Eqs. (95), have to be solved together with the BGR kinetic equations (78).

Acknowledgments

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